Applications of the Simplex Method

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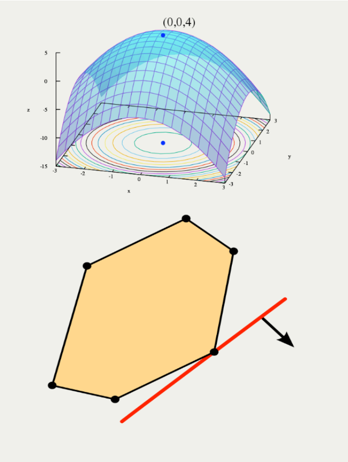
Quick Review

- Optimization technique
- Constraints in the form of inequalities
- Feasible Region
  - Intersection of 'half spaces'
  - Convex polytope
- Solution techniques
  - Graphically
  - Simplex Method



History

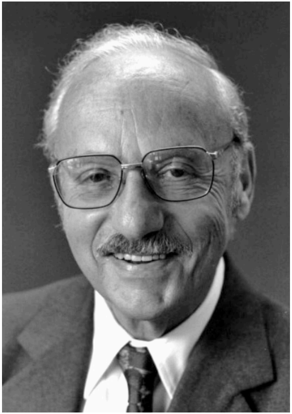


Figure: George Dantzig

Notoriously difficult to solve

- Joseph Fourier (1827): "Fourier-Motzkin" method
- Leonid Kantorovich (1939)
  - General formulation for linear programming problems
  - Expenditures and returns during WWII
- George Dantzig (1947): "Simplex Method"
  - Assigned 70 workers to 70 jobs

The General Linear Programming Problem

Objective Function:

$$z = c_1x_1 + c_2x_2 + ... + c_nx_n$$

Constraints of the form:

$$a_1x_1 + a_2x_2 + a_nx_n \begin{Bmatrix} \leq \\ = \\ \geq \end{Bmatrix} b$$

$$x_i \geq 0 \quad i \in \{1, ..., n\}$$

Definition

Feasible Solution: Any combination of variables such that all of the constraints are satisfied.

Full Standard Form

Maximize:

$$z = c_1 x_1 + c_2 x_2 + ... + c_n x_n$$

Subject to:

$$a_{11} x_{11} + a_{12} x_{12} + ... a_{1n} x_{1n} \leq b_1$$

$$\vdots$$

$$a_{m1} x_1 + a_{m2} x_2 + ... + a_{mn} x_n \leq b_n$$

$$x_i \geq 0 \text{ for } i \in 1, 2, ..., n$$

$$b_j \geq 0 \text{ for } j \in 1, 2, ..., m$$

Definition

Feasible Region: The set of all n-tuples that are feasible solutions

2-D Optimization

Maximize:

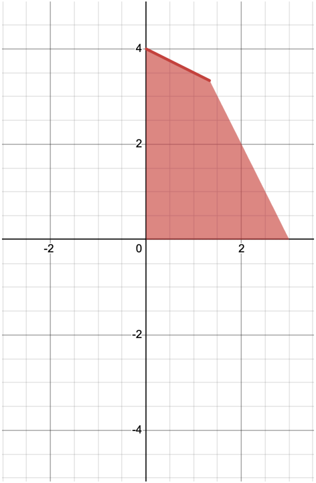
$$z = 3x_1 + 4x_2$$

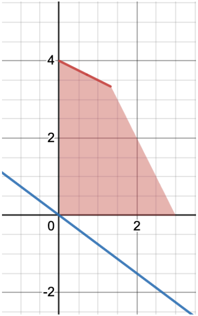
Subject to:

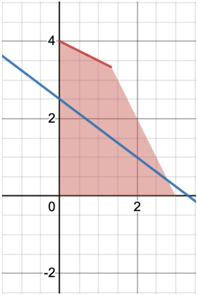
$$x_1 + 2x_2 \leq 8$$

$$2x_1 + x_2 \leq 6$$

$$x_{1,2} \geq 0$$




4
2
0
2
-2


4
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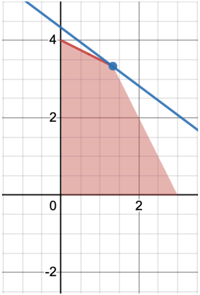

4
2
0
2
-2

Simplex Tableau vs Algebraic Method

Step 1:

basis	z	x_1	x_2	s_1	s_2	b
s_1	0	1	2	1	0	8
s_2	0	2	1	0	1	6
	1	-3	-4	0	0	0

Step 2:

basis	z	x_1	x_2	s_1	s_2	b
x_2	0	$\frac{1}{2}$	1	$\frac{1}{2}$	0	4
s_2	0	$\frac{3}{2}$	0	$-\frac{1}{2}$	1	2
	1	-1	0	2	0	16

Step 1: $x_1 = x_2 = 0$

$$s_1 = 8 - x_1 - 2x_2$$

$$s_2 = 6 - 2x_1 - x_2$$

$$z = 0$$

Step 2: Switch x_2 and s_1

$$x_2 = 4 - \frac{1}{2}x_1 - \frac{1}{2}s_1$$

$$s_2 = 2 - \frac{3}{2}x_1 + \frac{1}{2}s_1$$

$$z = 4(x_2) + 3(x_1) = 16 + x_1 - 2s_1$$

Solution

With one more iteration...

basis	z	x_1	x_2	s_1	s_2	b
x_2	0	0	1	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{10}{3}$
x_1	0	1	0	$-\frac{1}{3}$	$\frac{2}{3}$	$\frac{4}{3}$
	1	0	0	$\frac{5}{3}$	$\frac{2}{3}$	$17\frac{1}{3}$

$$x_1 = \frac{4}{3}$$

$$x_2 = \frac{10}{3}$$

$$z = 17\frac{1}{3}$$

Duality

What is the 'Dual' of L.P. in general form?

	Primal	Dual
variables	$x_1 \dots x_n$	$y_1 \dots y_m$
matrix	A	A^T
RHS	b	c
objective	max $c^T x$	min $b^T y$

The relationship between constraints is easier to picture in an example...

Example

Primal

Maximize:

$$z = 3x_1 + 4x_2$$

Subject to:

$$x_1 + 2x_2 \leq 8$$

$$5x_1 + 3x_2 \leq 6$$

$$4x_1 + 6x_2 \leq 3$$

$$x_{1,2} \geq 0$$

Dua1

Minimize:

$$z* = 8y_1 + 6y_2 + 3y_3$$

Subject to:

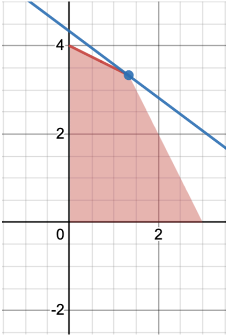
$$y_1 + 5y_2 + 4y_3 \geq 3$$

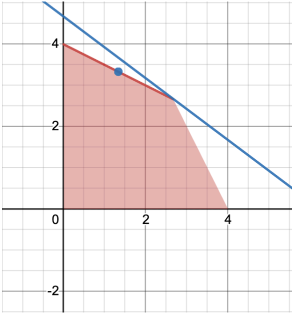
$$2y_1 + 3x_2 + 6y_3 \geq 4$$

$$y_{1,2} \geq 0$$

Why?

1. Sensitivity analysis
2. Establishes lower bound for Primal problem





*does not depict the previous slide

Problems

When we went to implement our own version of the simplex method, we ran into four problems we had to solve

1. Infeasible $\vec{0}$ Solution
2. Alternate Relational Operators ($=$, $\geq$)
3. Cycling

Infeasible $\vec{0}$ Solution

Guessing $\vec{0}$ as the basic feasible solution doesn't always work
Instead, we use the Two-Phase Simplex Method to find a basic feasible solution, then optimize the objective

Two-Phase Simplex Method

1. Add "Additional Variables" a_i to infeasible constraints
2. Ignoring the objective, minimize

$$\sum a_i$$

   - If $\sum_i a_i \neq 0$, the LP is infeasible
   - Otherwise, you've found a basic feasible solution
3. Remove the additional variables and solve the LP

Example

$$\begin{aligned} \text{maximize} \quad & 3x_1 + 4x_2 \\ \text{subject to} \quad & x_1 + 2x_2 \leq 8 \\ & 2x_1 + x_2 \leq 6 \\ & x_1 + x_2 \geq 2 \\ & x_{1,2} \geq 0 \end{aligned}$$

$$\begin{aligned} \text{maximize} \quad & 3x_1 + 4x_2 \\ \text{subject to} \quad & x_1 + 2x_2 + s_1 \leq 8 \\ & 2x_1 + x_2 + s_2 \leq 6 \\ & x_1 + x_2 - s_3 + a_1 \geq 2 \\ & x_{1,2} \geq 0 \end{aligned}$$

Example Cont.

Step 1: Make a Tableau

Step 2: Minimize

$$\sum a_i$$

Step 3: Run Simplex Iterations

basis	z	x_1	x_2	s_1	s_2	s_3	a_1	b
s_1	0	1	2	1	0	0	0	8
s_2	0	2	1	0	1	0	0	6
a_1	0	1	1	0	0	-1	1	2
z	1	-3	-4	0	0	0	0	0

basis	z	x_1	x_2	s_1	s_2	s_3	a_1	b
s_1	0	1	2	1	0	0	0	8
s_2	0	2	1	0	1	0	0	6
a_1	0	1	1	0	0	-1	1	2
z	1	0	0	0	0	0	1	0

basis	z	x_1	x_2	s_1	s_2	s_3	a_1	b
s_1	0	1	2	1	0	0	0	8
s_2	0	2	1	0	1	0	0	6
a_1	0	1	1	0	0	-1	1	2
z	1	-1	-1	0	0	1	0	-2

Example Cont.

Step 3 Cont.

basis	z	x_1	x_2	s_1	s_2	s_3	a_1	b
s_1	0	0	1	1	0	1	-1	6
s_2	0	0	-1	0	1	2	-2	2
x_1	0	1	1	0	0	-1	1	2
z	1	0	0	0	0	0	1	0

Step 4: Remove additional variables and plug in objective

basis	z	x_1	x_2	s_1	s_2	s_3	b
s_1	0	0	1	1	0	1	6
s_2	0	0	-1	0	1	2	2
x_1	0	1	1	0	0	-1	2
z	1	-3	-4	0	0	0	0

Step 5: Run Normal Simplex Iterations

basis	z	x_1	x_2	s_1	s_2	s_3	b
s_1	0	0	0	$\frac{1}{3}$	$\frac{1}{3}$	1	$\frac{8}{3}$
s_2	0	1	0	$-\frac{1}{3}$	$\frac{2}{3}$	0	$\frac{4}{3}$
a_1	0	0	1	$\frac{2}{3}$	$-\frac{1}{3}$	0	$\frac{10}{3}$
z	1	0	0	$\frac{5}{3}$	$\frac{2}{3}$	0	$\frac{52}{3}$

Optimal Solution: $x_1 = \frac{4}{3}$, $x_2 = \frac{10}{3}$

Alternate Operators

How can we deal with different relational operators?

- $(\leq)$ If $\sum_i a_{ij}x_i \leq b_j$, it becomes $\sum_i a_{ij}x_i + s_j = b_j$
- $(=)$ If $\sum_i a_{ij}x_i = b_j$, it becomes $\sum_i a_{ij}x_i + a_j = b_j$
- $(\geq)$ If $\sum_i a_{ij}x_i \geq b_j$, it becomes $\sum_i a_{ij}x_i - s_j + a_j = b_j$

Cycling

- Cycling is when you get to a previous tableau after performing simplex operations

Ex: If you pick the most negative value in the objective

basis	z	x_1	x_2	x_3	x_4	s_1	s_2	b
s_1	0	$\frac{1}{4}$	$-\frac{1}{8}$	12	10	1	0	0
s_2	0	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{5}$	0	1	0
z	1	-5	-4	20	2	0	0	0

$\Rightarrow$ After 6 Iterations

basis	z	x_1	x_2	x_3	x_4	s_1	s_2	b
s_1	0	$\frac{1}{4}$	$-\frac{1}{8}$	12	10	1	0	0
s_2	0	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{5}$	0	1	0
z	1	-5	-4	20	2	0	0	0

Bland's Rule

To stop cycling, we use Bland's Rule
Bland's Rule says we pick the first pivot of the possible choices in the objective

basis	z	x_1	x_2	x_3	x_4	s_1	s_2	b
s_1	0	$\frac{1}{4}$	$-\frac{1}{8}$	12	10	1	0	0
s_2	0	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{5}$	0	1	0
z	1	-5	-4	20	2	0	0	0

$\Rightarrow$

basis	z	x_1	x_2	x_3	x_4	s_1	s_2	b
x_2	0	2	1	1	14	20	0	0
s_2	0	$\frac{1}{2}$	0	$\frac{49}{4}$	$\frac{21}{1}$	$\frac{5}{2}$	1	0
z	1	3	0	24	18	80	0	0

Implementation

Using all of these, we made a Simplex Method implementation in Python/Numpy that uses Bland's Rule to stop cycling and can take any relation operation in its input

Scheduling Problem

- Three different types of employees
  - Managers ($25 per hour, 1.0 labor)
  - Regular workers ($18 per hour, 0.9 labor)
  - Trainees ($15 per hour, 0.5 labor)
- $b = \{3, 4, 5, 5, 4, 3, 3, 2\}$ Labor demand at hours (7am, ... 3pm)
- Constraints
  - Must have at least the minimum number of people necessary to cover demand
  - At least one manager working at all times
  - At least one trainee per day
  - At least as many regular workers as trainees
  - Each worker must work a 4 hour shift
- Minimize the cost of employees!

```
[[1.  0.9 0.5 0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0. ]
 [1.  0.9 0.5 1.  0.9 0.5 0.  0.  0.  0.  0.  0.  0.  0.  0. ]
 [1.  0.9 0.5 1.  0.9 0.5 1.  0.9 0.5 0.  0.  0.  0.  0.  0. ]
 [1.  0.9 0.5 1.  0.9 0.5 1.  0.9 0.5 1.  0.9 0.5 0.  0.  0. ]
 [0.  0.  0.  1.  0.9 0.5 1.  0.9 0.5 1.  0.9 0.5 1.  0.9 0.5]
 [0.  0.  0.  0.  0.  0.  1.  0.9 0.5 1.  0.9 0.5 1.  0.9 0.5]
 [0.  0.  0.  0.  0.  0.  0.  0.  0.  1.  0.9 0.5 1.  0.9 0.5]
 [0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  1.  0.9 0.5]
 [1.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0. ]
 [1.  0.  0.  1.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0. ]
 [1.  0.  0.  1.  0.  0.  1.  0.  0.  0.  0.  0.  0.  0.  0. ]
 [1.  0.  0.  1.  0.  0.  1.  0.  0.  1.  0.  0.  0.  0.  0. ]
 [0.  0.  0.  1.  0.  0.  1.  0.  0.  1.  0.  0.  1.  0.  0. ]
 [0.  0.  0.  0.  0.  0.  1.  0.  0.  1.  0.  0.  1.  0.  0. ]
 [0.  0.  0.  0.  0.  0.  0.  0.  0.  1.  0.  0.  1.  0.  0. ]
 [0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  1.  0.  0. ]
 [0.  0.  1.  0.  0.  1.  0.  0.  1.  0.  0.  1.  0.  0.  1. ]]
```

Solution

7am: 1 manager, 2 regular workers, 1 trainee

8am: 2 regular employees

9am: 2 regular employee

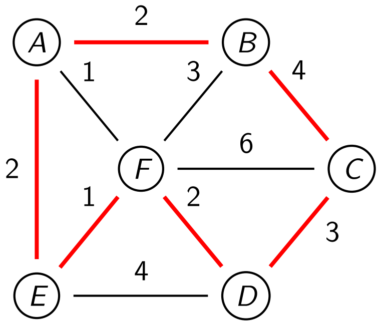
10am: nobody

11am: 1 manager, 2 regular employees

Cost: $\$200 + \$576 + \$60 = \$836$

Traveling Salesman

Given a graph with edge costs, find the least expensive cycle that visits every vertex



Naive Formulation

Let

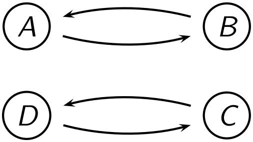
$$x_{ij} = \begin{cases} 1 \text{ if the edge from Node } i \text{ to Node } j \text{ is included} \\ 0 \text{ otherwise} \end{cases}$$

Then we want to

$$\begin{aligned} \text{minimize} \quad & \sum_{i=0}^{n} \sum_{j=0, j\neq i}^{n} c_{ij} x_{ij} \\ \text{subject to} \quad & \sum_{i=0, i\neq j}^{n} x_{ij} = 1 \qquad \forall j \quad \text{Outflow from each node is 1} \\ & \sum_{j=0, j\neq i}^{n} x_{ij} = 1 \qquad \forall i \quad \text{Inflow to each ndoe is 1} \\ & x_{ij} \in \{0, 1\} \quad \forall i, j \end{aligned}$$

Naive Formulation Cont.

This can get disconnected cycles, which doesn't solve the TSP



MTZ Formulation

To address this, we use the Miller–Tucker–Zemlin Formulation
For every $i \geq 2$, add a u_i to the Decision Variables with the constraint that $u_j > u_i$ if $x_{ij} = 1$.
In the program, this means

$$u_i - u_j + nx_{ij} \leq n - 1$$

Therefore, the actual LP Formulation should be

$$
\begin{aligned}
\text{minimize} \quad & \sum_{i=0}^{n} \sum_{j=0, j\neq i}^{n} c_{ij} x_{ij} & & \\
\text{subject to} \quad & \sum_{i=0, i\neq j}^{n} x_{ij} = 1 & \forall j & \quad \text{Outflow from each node is 1} \\
& \sum_{j=0, j\neq i}^{n} x_{ij} = 1 & \forall i & \quad \text{Inflow to each ndoe is 1} \\
& u_i - u_j + n x_{ij} \leq n - 1 & \forall i, j \geq 2 & \quad \text{MTZ Formulation} \\
& x_{ij} \in \{0, 1\} & \forall i, j &
\end{aligned}
$$

Example

For example, for a graph with cost matrix

	A	B	C	D
A		8	6	2
B	9		14	32
C	49	69		81
D	19	18	17	

The column is the origin and the row is the destination

Gets the cycle

$$A \to C \to B \to D \to A$$

Minimize:

```
[ 9 49 19  8 69 18  6 14 17  2 32 81  0  0  0]
```

$\vec{x}$

Subject To:

```
[[0. 0. 0. 1. 0. 0. 1. 0. 0. 1. 0. 0. 0. 0. 0.]
 [1. 0. 0. 0. 0. 0. 0. 1. 0. 0. 1. 0. 0. 0. 0.]
 [0. 1. 0. 0. 1. 0. 0. 0. 0. 0. 0. 1. 0. 0. 0.]
 [0. 0. 1. 0. 0. 1. 0. 0. 1. 0. 0. 0. 0. 0. 0.]
 [1. 1. 1. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 1. 1. 1. 0. 0. 0. 0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0. 1. 1. 1. 0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 1. 1. 0. 0. 0.]]
```

$\vec{x} =$

```
1.0
1.0
1.0
1.0
1.0
1.0
1.0
1.0
```

```
[[ 0.  0.  0.  0.  4.  0.  0.  0.  0.  0.  0.  0.  1. -1.  0.]
 [ 0.  0.  0.  0.  0.  4.  0.  0.  0.  0.  0.  0.  1.  0. -1.]
 [ 0.  0.  0.  0.  0.  0.  0.  4.  0.  0.  0.  0. -1.  1.  0.]
 [ 0.  0.  0.  0.  0.  0.  0.  0.  4.  0.  0.  0.  0.  1. -1.]
 [ 0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  4.  0. -1.  0.  1.]
 [ 0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  4.  0. -1.  1.]]
```

$\vec{x} \leq$

```
3
3
3
3
3
3
```

Gets:

```
[0. 1. 0. 0. 0. 1. 0. 1. 0. 1. 0. 0. 2. 0. 3.]
```

Summary

What we did:

- Build a robust simplex implementation
- Look at two applications
  - Scheduling Problem
  - TSP

What we're working on:

- More examples
- Write up